THE LOGISTIC DIFFERENTIAL EQUATION AND THE BIFURCATION OF THE QUADRATIC FAMILY

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ABSTRACT

Using the logistic differential equation, the logistic difference equation is obtained. We then use numerical analysis to investigate this difference equation, setting it up for repeated iterations. It is then shown that the logistic differential equation is related to the quadratic family, which is then analysed both numerically and graphically to demonstrate the phenomenon of bifurcation.

INTRODUCTION

The logistic equation is an important equation when modeling population growth, as it models a bounded population subject to different constraints. By varying the constraints, we discover bifurcation. Bifurcation, without spoiling the surprise, pops up in *many* applications different fields. Although it is not discussed further in the paper, the bifurcation diagram appears in fields such as physics, when it crops up when analysing the motion of dampened springs. In fact the entire field of chaos has a wide-range of applications, and unfortunately only a little glimpse is provided here.

THEORY

The logistic differential equation

$$\frac{dP}{dt} = aP - bP^2 \qquad (a, b > 0) \tag{1}$$

is a basic model that can describe the behavior of some species population $P(t)$ that is subject to no external influences. Assuming that the actual population of the species can be determined for the current generation, it will be seen that the population at some time n can be predicted solely from the knowledge of the present population. This differential equation can be solved using separation of variables.

$$\frac{dP}{dt} = aP - bP^2$$

$$\frac{dP}{aP - bP^2} = dt$$

$$\frac{1}{aP(1 - \frac{b}{a}P)} dP = dt$$

$$\left(\frac{A}{aP} + \frac{B}{1 - \frac{b}{a}P}\right) dP = dt$$

Taking the case where $a = b = 1$ and using Partial Fraction Expansion (PFE) we get

$$A = \left.\frac{1}{1 - P}\right|_{P=0} = 1$$

$$B = \left.\frac{1}{P}\right|_{P=1} = 1$$

$$\Downarrow$$

$$\int \left(\frac{1}{P} - \frac{1}{P-1} \right) dP = t$$

$$t = clnP - cln(P-1)$$

$$ce^t = \frac{P}{P-1}$$

$$P(ce^t - 1) - ce^t = 0$$

$$\Downarrow$$

$$P = \frac{ce^t}{ce^t - 1}$$

This indicates that the logistic differential equation is a bounded one, which is why it is a good simple model for population growth. It is desired, however, to use the computational power that is available and pursue a numerical analysis, illustrating one use of Euler's Method.

As in the section describing Euler's Method, we must first pick a step size h, which designates the time between times $t_0, t_1, t_2, \ldots, t_n, \ldots$ such that $t_{n+1} = t_n + h$. In the case of population modeling, h might represent the time between mating seasons, for example. Given this, we see that by Euler's Method we wish to approximate values of $P_1, P_2, P_3, \ldots, P_n, \ldots$ by calculating iterations of the following:

$$P_{n+1} = P_n + (aP_n - bP_n^2) \cdot h \tag{3a}$$

Factoring out a P_n and distributing the h, we see that we have

$$P_{n+1} = P_n + P_n(ah - bhP_n)$$

$$\Downarrow$$

$$P_{n+1} = P_n(1 + ah - bhP_n)$$

Now we can let $1 + ah = c$ and $bh = d$, reducing the equation to

$$P_{n+1} = cP_n - dP_n^2 \tag{3b}$$

One final substitution of $P_n = \frac{c}{d}x_n$ yields Equation 3c:

$$x_{n+1} = cx_n(1 - x_n) \tag{3c}$$

This is a simplified form of the logistic difference equation, which was derived from the logistic differntial equation using Euler's method. The beauty of the logistic difference equation is that given an initial population x_0, sucessive x_n's may be calculated:

$$x_1 = cx_0(1 - x_0)$$
$$x_2 = cx_1(1 - x_1)$$
$$\vdots$$
$$x_n = cx_{n-1}(1 - x_{n-1})$$

Of course, actual population modeling is extremely complicated, but by analyzing the end-behavior of the orbits produced by Equation 3c, the fate of any initial population can be determined. Although the logistic function is a fairly basic equation, the math can get rather complicated when computing orbits' fate. For this reason, we turn our attention to a function closely related to the logistic function- that of the quadratic family. It can be seen from Equation 3b that if we let $c \to 0^+$ we will have

$$P_{n+1} = 0^+ - dP_n^2$$

Thus, by letting one constant → 0, we have changed the logistic difference equation into a pure quadratic expression. Thus it is not surprising that both the logistic difference equation and the quadratic that will be analysed in Equation 4a display similar bifurcation and orbit behavior.

As mentioned above, the specific quadratic family that will be analysed is

$$F_c(x) = x^2 + c \tag{4a}$$

As with the logistic function, given an initial P_0 we can determine the 'population' for any generation n, (I simply substituted P_n for x_n).

$$
\begin{aligned}
P_1 &= P_0^2 + c \\
P_2 &= P_1^2 + c \\
&\vdots \\
P_n &= P_{n-1}^2 + c \qquad (4b)
\end{aligned}
$$

So now we concern ourselves with how $F_c(x) = x^2 + c$ is affected by the parameter c; specifically, we are interested in how the fate of the population, i.e.

$$x_\infty = \lim_{n \to \infty} x_n \qquad (5)$$

is affected by c and to accomplish this, we will be investigating the fate of various orbits.

MEASUREMENTS AND SIMULATION RESULTS

Thus as stated above, we ask what values of c yield interesting results- and what are these results? We can graph the function $F_c(x) = x^2 + c$ along with the quadratic function $G(x) = x$ in order to graphically analyze this. Figure 1 illustrates the case of $F_c(x) = x^2 + c$ for $c > .25$, and Figure 2 illustrates the case of $F_c(x) = x^2 + c$ for $c = .25$. Iterating Equation 4a amounts to taking an initial x value, plugging it into the equation, then taking that result for use as the new x value. Graphically, this can be done by taking an initial x value along the graph of $G(x) = x$ and drawing a line either up or down until it meets $F_c(x) = x^2 + c$, in effect evaluating Equation 4a at the point selected. Now, draw a line left or right until it itersects $G(x) = x$. This is the new point to use, and we have completed one iteration. It can be seen that for Figure 1, any point selected as the initial value will eventually approach infinity with repeated iterations, or

$$F_c(x) = x^2 + c \Rightarrow \infty \qquad (c > .25) \qquad (6)$$

Intuitively, the graph of $F_c(x) = x^2 + c$ must at least intersect $G(x) = x$ in order for the points to have any chance of settling down, otherwise we have the condition in Equation 6. This can be shown mathematically. These graphs are equal when

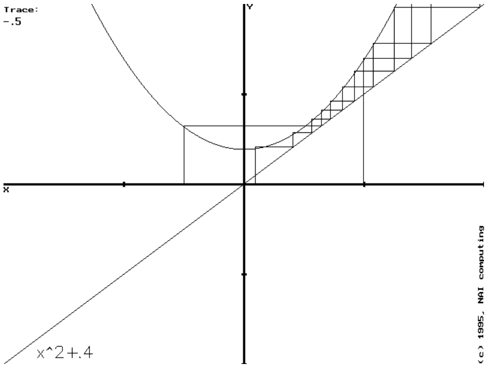

Figure 1: Iterations of $x^2 + .4$

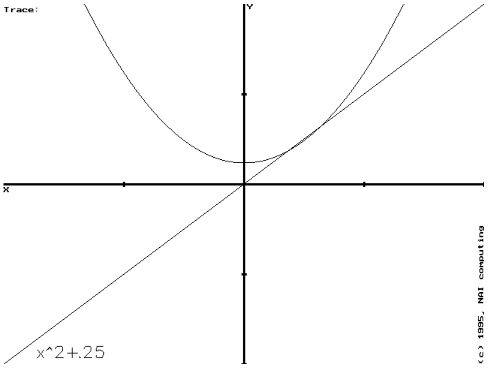

Figure 2: Iterations of $x^2 + .25$

$$F_c(x) = G(x)$$
$$x^2 + c = x$$
$$x^2 - x + c = 0$$

This is an ordinary quadratic equation, and can be solved using the quadratic formula. Doing so, we see that the roots of this equation are

$$a = \frac{1 + \sqrt{1 - 4c}}{2} \qquad (7a)$$

$$b = \frac{1 - \sqrt{1 - 4c}}{2} \qquad (7b)$$

and the points of intersection are (a, a) and (b, b). Of course this means that we will only have intersections when the $\sqrt{1 - 4c}$ term is positive, or

$$1 - 4c \geq 0$$
$$c \leq \frac{1}{4}$$

This is where the upper bound of .25 comes from. If the value of c is taken to be less than .25, say for example $c = -.5$, we have a graph resembling Figure 3. Using Equations 7a and 7b, we see that the points of intersections are $(1.366, 1.366)$ and $(-.366, -.366)$. Using the program *trace.pas* to graphically calculate the iterations, Figure 3 shows that for initial values of x such that $-1.366 \geq x$ and $1.366 \leq x$, we see that the orbits escape to infinity- Equation 6! However, we also see that when x is within these bounds, the iterations are attracted to the point $(-.366, -.366)$. This is known as an *attracting fixed point*. When $c = -.5$ and $-1.366 \leq x \leq 1.366$, sucessive iterations will yield $x = -.366$, a point which is verified by Figure 3. This is shown in Table 1 using values from the program *iterate.cc*.

At this point it only seems logical to make a plot of c versus the end behavior of each orbit as the number of iterations of x increases. The program *orbits.pas* does this by dividing the x-axis into 640 equally spaced values of c from some $c1$ to $c2$. Each pixel's width Δh is such that $\Delta h = \frac{c1-c2}{640}$. The y-axis, then, represents the end-behavior for each c, appropriately scaled. The first 100 iterations are thrown

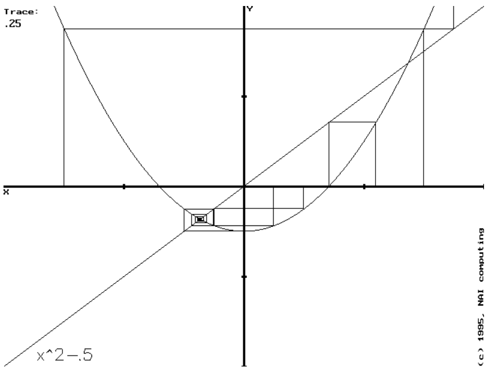

Figure 3: Iterations of $x^2 - .5$

x	10	1000	100000
1.36	-.3156	-.3660	-.3660
.9	-.3692	-.3660	-.3660
.3	-.3632	-.3660	-.3660
-.3	-.3632	-.3660	-.3660
-.9	-.3692	-.3660	-.3660
-1.36	-.3156	-.3660	-.3660

Table 1: Iterations of x, $c = -.5$

out as x settles and iterations 100-200 are plotted. The output is the "pitchfork" diagram shown in Figure 4. The values of c are $-2 \leq c \leq .25$.

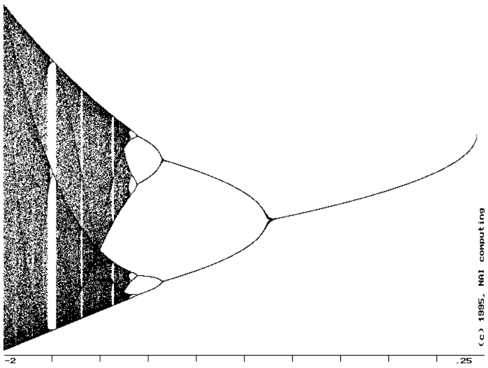

Figure 4: The "Pitchfork" Diagram

What is this? Since the line represents end behavior of x or each orbits fate, we see that for our previous example of $c = -.5$, we see that the orbits settle down to a single value, which we found to be $x = -.366$. But what then is this division of our line and why and where does it occur? This is known as *bifurcation* and it shows that instead of x being attracted to a single point, that is, x is settling down to a single value, x is now alternating between two different end-points- the period has doubled! In other words, x will ultimately jump between two final points, with every other iteration yielding the same final value for x, just as

$$A_{x+1} = (-1) \cdot A_x$$

will alternate between -1 and 1 if $A_0 = -1, 1$. To investigate this further, we need to look at the math.

We know that these points are solutions to $F_c^2(x) + c = x$ or

$$(x^2 + c)^2 + c = x$$

or

$$x^4 + 2cx^2 - x + c^2 + c = 0 \tag{8}$$

Our job is to find the roots of this equation. However we know that Equations 7a and 7b are solutions of $F_c(x) = x^2 - x + c$, so we can just divide Equation 8 by this.

$$\begin{array}{r} x^2 + x + c + 1 \\ x^2 - x + c\sqrt{x^4 + 2cx^2 - x + c^2 + c} \end{array}$$

As can be seen, this will yield the equation

$$x^2 + x + c + 1 \tag{9}$$

Which, using the quadratic equation, has roots of

$$c = \frac{-1 + \sqrt{1 - 4(c+1)}}{2} \tag{10a}$$

$$d = \frac{-1 - \sqrt{1 - 4(c+1)}}{2} \tag{10b}$$

Of course, this will only yield real numbers when $1 - 4(c+1) \geq 0$ or

$$c \leq -\frac{3}{4}$$

This shows us that this doubling of orbit cycles occurs when $c \leq -.75$; the orbits of x will alternate between two values. Let us check the value of $c = -.75$ both by tracing the iterations and analyzing the orbit diagram. Figure 5 shows an enlargement of Figure 4, with $-.65 \leq c \leq -.85$. The bifurcation of the "pitchfork" is seen to occur in the middle, at $c = -.75$. Figure 6 shows graphically the end-behavior of x when $c = -.8$. It is hard to tell exactly what is occuring here, so in Figure 7 the first 1000 iterations were not plotted, thus giving the ultimate behavior of x. It can be seen that x does indeed alternate between 2 values. The values in Table 2 were obtained from *iterate.cc* and match exactly the numbers produced by using Equations 10a and 10b with $c = -.8$.

x	10	100000
.5	-.561, -.485	-.724, -.276
.8	-.245, -.739	-.724, -.276
-.9	-.237, -.744	-.724, -.276

Table 2: Iterations of x, $c = -.8$

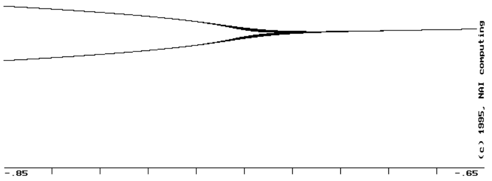

Figure 5: Bifurcation for $-.85 < c < -.65$

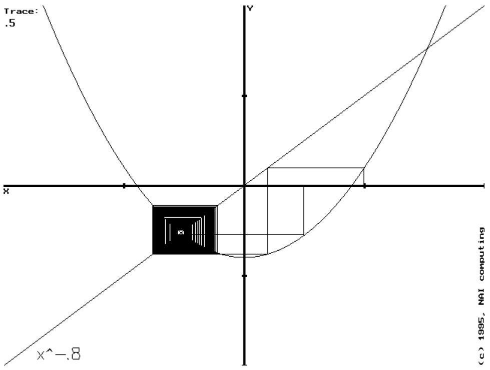

Figure 6: Iterations of $x^2 - .8$

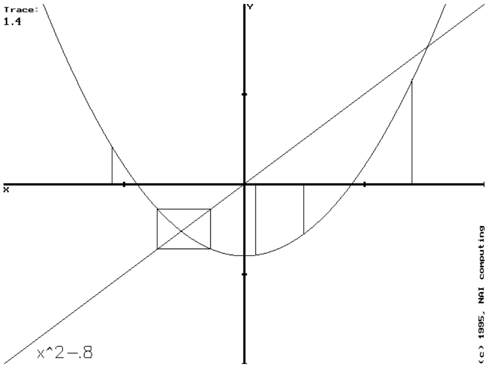

Figure 7: Iterations of $x^2 - .8$

However, the convention of using pen-and-paper to solve for the points of bifurcation soon gets very impractical. The reason for this is that to find the n-th bifurcation point, we must solve a polynomial equation of the order 2^n. Since it can be proved that there is no general method for solving polynomial equations of degree greater than 4 (Devaney), it is almost impossible to any further calculations without the use of the computer. Thus with this known, the same procedure can be repeated for the next bifurcation, when the period doubles again to a period of four. Now x settles into an orbit consisting of four values. Every four iterations, a value is repeated. This is shown in Figure 8, an enlargement of Figure 4, with $-1.35 \leq c \leq -1.15$ The two branches that represented an orbit of period two have bifurcated, and an orbit of period four emerges. This orbit is clearly seen in Figure 9, with the first 1000 iterations not plotted. Table 3's data clearly shows the period increasing to 4.

x	100000
1	-1.15,0.019,-1.29,0.389
-.5	-1.15,0.019,-1.29,0.389
-.01	-1.15,0.019,-1.29,0.389

Table 3: Iterations of x, $c = -1.3$

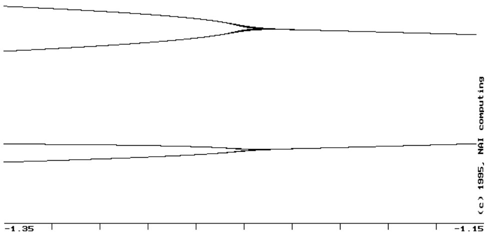

Figure 8: Bifurcation for $-1.35 < c < -1.15$

Finally, we repeat the previous work with the next bifurcation point, when the orbit changes from a pe-

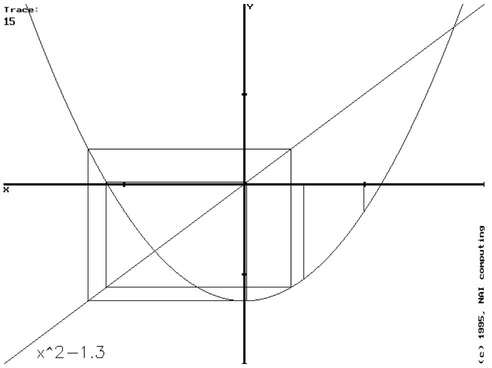

Figure 9: Iterations of $x^2 - 1.3$

riod 4 to a period 8. From the pitchfork diagram, Figure 4, we can see that the next bifurcation point occurs around $c = -1.36$. Extensive computer calculations yield a precise answer of $c = -1.365$, which is verified by Figure 10. Thus, we use $c = -1.37$ so that the new period is sufficiently defined. The data is recorded in Table 4, and the new period of 8 can clearly be seen in Figure 11.

x	100000
-.5	-1.12,-.011,-1.36,.47
	-1.14,-.05,-1.37,.50
1.1	-1.12,-.011,-1.36,.47
	-1.14,-.05,-1.37,.50

Table 4: Iterations of x, $c = -1.37$

Finally, it is worth mentioning some of the interesting aspects of pitchfork diagram. Since the black represents where the iterations of x fall, we can see that the iterations divide and divide, yielding the left-hand side of Figure 4. If we let $c = -2$ where this is the worst, and plot only the last 100 iterations of 1000, we see that the pattern has become chaotic- this is Figure 12.

There is obviously no clear 2, 4 or 8 cycle here. The reason why the bifurcation has become chaotic is more than enough for another paper, but the importance is that the system has gone from an orderly subdivision of orbits to utter chaos. Chaos, however,

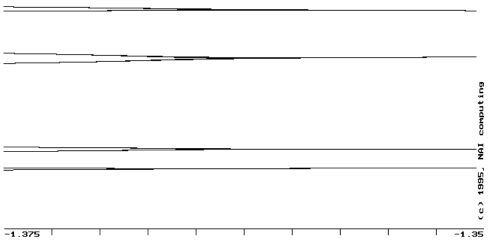

Figure 10: Bifurcation for $-1.375 < c < -1.355$

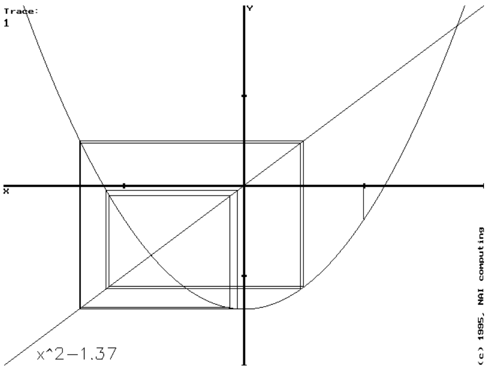

Figure 11: Iterations of $x^2 - 1.37$

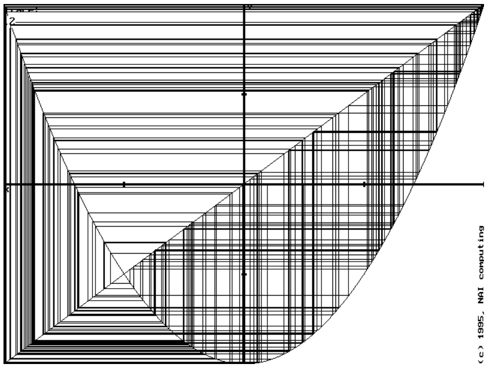

Figure 12: Iterations of $x^2 - 2$

rules in other parts of Figure 4 and one of the most interesting facets of the diagram is when the orbits *return* to order from the chaos. Figures 13, 14 and 15 are enlargements of areas that incredibly return from incredibly high orbits or chaos to simple 4 or 8 period cycles. Not only do they return, but they return and begin the initial bifurcation again, from 2, 4, etc. They are, in effect, miniature bifurcation graphs within the larger bifurcation graph. There is much more to discuss and analyse, yet the basic concepts have at least been touched upon.

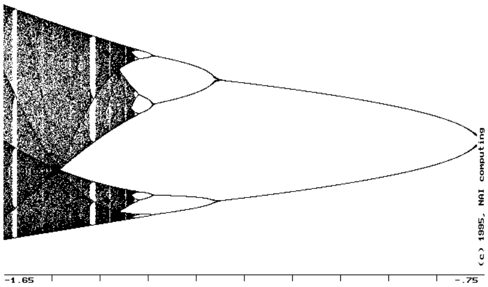

Figure 13: Bifurcation for $-1.65 < c < -.75$

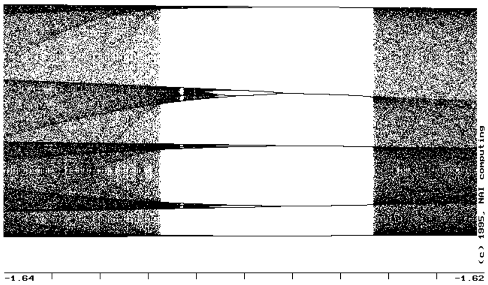

Figure 14: Bifurcation for $-1.64 < c < -1.62$

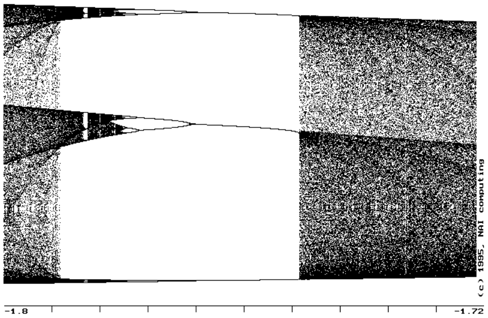

Figure 15: Bifurcation for $-1.8 < c < -1.72$

CONCLUSION

In conclusion, we have seen how the logistic differential equation can be represented as the logistic difference equation. This equation is linked to the equations of quadratic family, and by analysing this family, we were able to see behavior common to both. The study of bifurcation is a large and complex field of complex mathematics, whether the reader knows it or not. By allowing the graph to include complex numbers, and then plotting the bifurcation of the quadratic family in the complex plane, we stumble upon the famous Mandelbrot Set and fractals, whose behavior itself is *closely* linked to the behavior studied in this paper. It is a fascinating field, and it truely stems from the study of differential equations.